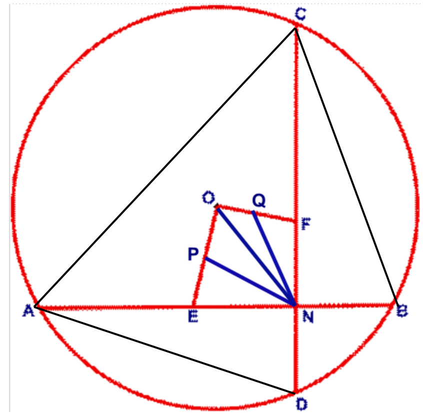
01.01.2024 Cash Award Math Rider - Author's Solution

To Prove: $(OE^2 - OF^2) = 2(NE^2 - NF^2)$.



Construction:

Let P & Q be the midpoints of OE & OF respectively.

Join PN, QN, BC, AC & AD.

As FN= ND, F is the orthocentre of ΔABC.

As EN = NB, E is the orthocentre of ΔADC.

And O is the circumcentre.

As Centre of nine-point circle is the midpoint of the segment joining the orthocentre and the circumcentre of a triangle,

P is the centre of nine-point circle of ΔADC & Q is the centre of nine-point circle of ΔABC.

As radius of nine-point circle is half of the radius of circumcircle and here circumcircle of two triangles ΔABC & ΔADC are same.

$\Longrightarrow$ Radius of two nine-point circle are equal. Also both nine-point circle passes through N (as N is the foot of the altitudes of both the triangles ΔABC & ΔADC and nine-point circle always passes through the foot of the altitudes)

$\Longrightarrow$ PN = QN = radius of nine-point circle.

In ΔONE, NP is the median.

So, by Apollonius Theorem,

$ON^2 + NE^2 = 2\left(\frac{OE^2}{4} + PN^2\right)$ -------------------------(1)

In ΔONF, NQ is the median.

So, by Apollonius theorem.

$ON^2 + NF^2 = 2\left(\frac{OF^2}{4} + NQ^2\right)$ --------------------------(2)

(1) - (2)

$NE^2 - NF^2 = \frac{OE^2}{2} - \frac{OF^2}{2}$ $\quad (\because PN = NQ)$

$\Longrightarrow 2(NE^2 - NF^2) = OE^2 - OF^2$ ------------------------ Proved

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